Grid Generation Techniques in Computational Fluid Dynamics

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Introduction

NUMERICAL grid generation has now become a fairly common tool for use in the numerical solution of partial differential equations on arbitrarily shaped regions. This is especially true in computational fluid dynamics, from whence came much of the impetus for the development of this technique, but the procedures are equally applicable to all physical problems that involve field solutions. This review, and the other general discussions cited, are relevant to the numerical solution of field problems at large, although most of the specific applications noted are from computational fluid dynamics.

Numerical grid generation is basically a procedure for the orderly distribution of observers over a physical field in a way that efficient communication among the observers is possible and all physical phenomena on the entire continuous field may be represented with sufficient accuracy by this finite collection of observations. This technique frees the computational simulation from restriction to certain boundary shapes and allows general codes to be written in which the boundary shape is specified simply by input. The boundaries may also be in motion, either as specified externally or in response to the developing physical solution. Similarly, the observers may adjust their positions to follow gradients developing in the evolving physical solution. In any case, the numerically generated grid allows all computation to be done on a fixed square grid in the computational field. (Computational field refers to the space of the curvilinear coordinates, i.e., where these coordinates serve as independent variables, rather than the Cartesian coordinates. This field is always rectangular by construction, as explained in Ref. 184.)

The area of numerical grid generation is relatively young in widespread practice, although its roots in mathematics are old. This area involves the engineer's feel for physical behavior, the mathematician's understanding of functional behavior, and a lot of imagination, with perhaps a little help from Urania. The physics of the problem at hand must ultimately direct the grid points to congregate so that a functional relationship on these points can represent the physical solution with sufficient accuracy. The mathematics controls the points by sensing the gradients in the evolving physical solution, evaluating the accuracy of the discrete representation of that solution, communicating the needs of the physics to the points, and, finally, providing mutual communication among the points as they respond to the physics.

The basic techniques involved then are 1) a means of distributing points over the field in an orderly fashion, so that neighbors may be easily identified and data can be stored and handled efficiently; 2) a means of communication between points, so that a smooth distribution is maintained as points shift their positions; 3) a means of representing continuous functions by discrete values on a collection of points with sufficient accuracy, and a means for evaluation of the error in this representation; and 4) a means for communicating the need for a redistribution of points in the light of the error evaluation, and a means of controlling this redistribution.

It should be borne in mind that the requirements, e.g., smoothness, orthogonality, etc., that must be met by the grid are ultimately determined by the numerical algorithm to be run on the grid. Thus, at the same time that effort is made to generate better grids, a similar effort should be made to develop hosted algorithms that are more tolerant of the grids.

Considerable progress has been made in the past decade, especially in the last few years, toward the development of these techniques and toward casting them in forms that can be readily applied. A comprehensive survey of procedures and applications through 1981 has been held,[190] and two conferences specifically on the area of numerical grid generation have been held.[167,186] Some expository papers are included in the latter proceedings,[186] which can serve as an introduction to the area. Brief surveys of numerical grid generation in computational fluid dynamics are included in Refs. 192, 116, and 104.

The present review attempts primarily to make some correlations among the various approaches that have been taken, and thus to provide some more clearly defined avenues for present application and future effort. Individual references to papers cited in Ref. 190 are not made here as points covered in that survey are noted. Some significant advances and applications have been reported during the past year, and these are reviewed here in relation to earlier work. Particular attention is given to grid-induced error, dynamically adaptive grids, and complicated configurations. Standard notation for the metric coefficients is used. A complete definition of the relations involved with general curvilinear coordinate systems is given in Ref. 184.

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Some Basic Features

Throughout this review, and in the references cited, a numerically generated grid is understood to be the organized set of points formed by the intersections of the lines of a boundary-conforming curvilinear coordinate system. The cardinal feature of such a system if that some coordinate line (surface in three dimensions) is coincident with each segment of the boundary of the physical region. This allows boundary conditions to be represented entirely along coordinate lines without need of interpolation. The use of coordinate line intersections to define the grid points provides an organizational structure which allows all computation to be done on a fixed square grid when the partial differential equations of interest have been transformed so that the curvilinear coordinates replace the Cartesian coordinates as the independent variables.

The basic ideas of the generation of such grids, the necessary transformation relations, and the procedures for application in the numerical solution of partial differential equations are assembled in an introductory fashion in Ref. 184. The various types of boundary-conforming coordinate systems, and the different methods of numerical generation thereof, are discussed in some detail in Ref. 190. Numerous examples of applications to field problems are also cited in this reference. The subject of numerical grid generation is still too young for complete evaluation of the different approaches. However, some directions are emerging and some relative evaluations are given in Ref. 190, as well as in the present review. Different types of grids are more appropriate to different physical problems and configurations, and also to different modes of usage. The following excerpt from Ref. 48 sums up the general situation well:

> To choose between the various types of coordinates, we must first consider which constraints are needed for a given problem. The fundamental constraint for a general region is its boundary geometry. When the coordinates match the boundary, the need for boundary interpolation disappears and the grid is also aligned with the desired solution near the boundary. Without any further requirement in the two-dimensional case, conformal systems are usually the best. In addition to boundary geometry, however, the pointwise distribution along the boundary is often required as a further constraint. This distribution is a boundary coordinate system or systems, which together with the geometry forms a complete boundary representation. When the representation is arbitrarily prescribed, conformal transformations are not applicable because of analytic continuation. As the next simplest case, orthogonal coordinates are preferable. In two dimensions they are generally applicable on both planes and curved surfaces. In three-dimensional regions, orthogonal systems are severely restricted and are not generally applicable. The best that can be done in the general context is to bound such regions with orthogonal systems so that full orthogonality can be specified at the boundaries. Further boundary constraints can also be imposed with specified derivatives so that rates of entry or exit from a region can be given. In any dimension, the capability to create a smoothly assembled composite mesh for topologically complex configurations would be achieved. In addition to the various boundary contraints, significant advantages can be obtained under the constraint that points, curves or surfaces be clustered in some location within the region. The purpose is usually to more fully resolve the numerical solution of a given problem with a fixed number of mesh points. Additional advantages can also be achieved with the constraint that a certain desirable mesh structure be smoothly embedded within the region.

Coordinate systems that are othogonal, or at least nearly orthogonal, near the boundary make the application of boundary conditions more straightforward. Although strict orthogonality is not necessary, and conditions involving normal derivatives can certainly be represented by difference expressions that combine one-sided differences along the line emerging from the boundary with central expressions along the boundary, the accuracy deteriorates if the departure from orthogonality is too large, as discussed more fully in a later section. It may also be more desirable in some cases not to involve adjacent boundary points strongly in the representation, e.g., on extrapolation boundaries. The implementation of algebraic turbulence models is more reliable with near-orthogonality at the boundary, since information on local boundary normals is usually required in such models. The formulation of boundary-layer-type equations is also much more straightforward and unambiguous in such systems. Similary, as noted for instance in Ref. 10, algorithms based on the parabolic Navier-Stokes equations require that coordinate lines approximate the flow streamlines and the lines normal thereto, especially near solid boundaries. It is thus better, in general, other considerations being equal, for coordinate lines to be nearly normal to boundaries.

Configurations

Much effort is now being directed toward devising appropriate configurations of the computational field for complicated physical regions, expecially in three dimensions. The introduction of branch cuts allows general physical regions to be transformed to rectangular blocks in the computational field. It is also possible to embed computational subfields in the overall computational field through cuts in the form of slits in the field. The use of branch cuts involves continuous passage onto different Riemann sheets, and some basic considerations in formulating difference representations across such cuts are given in Ref. 184. Further discussion of branch cuts is given in Ref. 185 with particular regard to the treatment of the control functions used in elliptic generation systems.

Although in principle any region can be transformed into an empty rectangular block through the use of branch cuts, the resulting grid point distribution may not necessarily be reasonable in the entire region. Furthermore, an unreasonable amount of effort may be required to properly segment the boundary surfaces and to devise an appropriate point distribution thereon for such a transformation. Some configurations are better treated with a computational field that has slits or rectangular holes in it. Reference 76 gives a procedure for conformally mapping a multibody configuration to a group of rectangular blocks in the field. The use of different configurations of the computational region is covered in a basic manner in Ref. 184, and examples are given in Ref. 190.

A code called INMESH, for the generation of two-dimensional grids on different types of configurations in the transformed region, including the use of slits and blocks, from an elliptic generation system, is described in Ref. 34. Another two-dimensional code, WESCOR,[183] of the same basic type, but which also includes control functions for attraction of coordinate lines and/or points to other grid lines or points, or to fixed lines or points, has also been written.

Configurations particularly appropriate for cascades are discussed in Refs. 95, 96, and 42. The first two references favor the slit type (called H-type there) while the last favors the C-type. The C-type gives better treatment of the leading-edge region since it wraps around the airfoil, but is very sparse and badly skewed far upstream, especially for highly staggered and cambered blades. Concentration of coordinate lines near the blade is also more efficient with the C-type. The advantage of the H-type is its better treatment of the far upstream region. Concentration near the blade is possible with this type, but points are wasted upstream. There is also a point on the leading edge which requires special treatment. Further discussions of cascade configurations are given in Ref. 190.

Patched Subregions

More complicated bodies, such as typical aircraft configurations, are better treated by generating grids for contiguous subregions which are then patched together to cover the entire physical region. This approach is discussed in Ref. 153. Reference 110 also uses this approach, as does Ref. 151. This latter reference discusses some automated procedures for generating such grids. Further discussion also appears in Ref. 190. Reference 136 uses subregion grids for a case in which the subregion interfaces are actually material boundaries. The subregion approach has been used recently for cascade configurations in Ref. 63.

Boundaries of such subregions that are not physical boundaries require that the coordinate lines have sufficient continuity across the boundary, or else one-sided difference representations would have to be used thereon. The GRAPE code of Ref. 172 generates a two-dimensional grid with control of the intersection angle on the boundaries, using an elliptic generation system. This is accomplished by iteratively adjusting the control functions on the boundaries, with exponential interpolation of the control functions between the boundaries. This approach can be used to generate subregion grids with continuous coordinate line slopes across the subregion boundaries. Further discussion and an application to viscous compressible flow appear in Ref. 106. This approach has recently been used in Ref. 155 and a similar approach has recently been used in Ref. 161, for which applications to transonic potential flow are given in Ref. 162. In Ref. 106 the subregion grids are made to overlap in order to aid in the representation of boundary conditions on the interfaces, but still with no interpolation. Here some restriction on the time step is required when shocks intersect the subregion interfaces.

Reference 107 also gives a procedure in which the intersection angle at the boundary is controlled in an elliptic generation system by iteratively adjusting one control function at the boundary, in this case using linear interpolation for the control function in the field. No iterative spacing control is incorporated, however. The approach of the GRAPE code recently has been extended to three dimensions in Ref. 173. A similar three-dimensional approach has also been given recently in Ref. 58, but with the transformation taken from a spherical coordinate system.

Control of the intersection angle on boundaries can also be achieved by increasing the order of the elliptic generation system so that additional boundary conditions are allowed. In Ref. 166, the biharmonic equation, which is fourth order, is used to generate subregion grids which can be joined with continuous coordinate line slopes.

In Ref. 73, the segmentation approach is applied to multiple-body configurations using conformal mapping. There each subregion contains a single body and transforms into an annulus. There are, however, sparse areas near the subregion corners. Some improvement is gained by combining this subregion approach with the string mapping (where all bodies are connected in a string), noted below in the conformal section.

Subregion grids can also be joined with continuity of coordinate lines and slope using algebraic grid generation systems having sufficient degrees of freedom in the interpolation, e.g., Hermite or spline interpolation. A grid embedding procedure based on the multisurface method, discussed later, is given in Refs. 49 and 50. Transfinite interpolation, also discussed below, can also be used with specified line slopes so that patching can be done with slope continuity.

Reference 80 gives an Euler equation solution for transonic flow using the subregion approach with slope continuity on the interface. Here careful attention must be given to flux balancing at the subregion interfaces, using transition operators to change from central to one-sided differences. Comparisons with single-region solutions show no degradation in accuracy or convergence, even when shocks propagate through the interfaces.

General three-dimensional regions can be built up using subregions as follows: First, point distributions are specified on the edges of a curved surface forming one boundary of a subregion, and a two-dimensional coordinate system is generated on the surface. When this has been done for all surfaces bounding the subregion, the three-dimensional system within the subregion is generated using the surface grids as boundary conditions. This type of segmentation approach is discussed in Refs. 180, 190, 153, and 185, with the first reference giving considerable detail. This subregion approach is probably the most effective available at present for general three-dimensional configurations.

In using this approach with elliptic generation systems, the control functions for the two-dimensional surface solution can be determined by evaluating one-dimensional control functions from the specified point distribution on the edges and then interpolating over the surface. Similarly, the two-dimensional control functions can be evaluated from the resulting surface solutions and interpolated into the three-dimensional field. This approach allows some degree of indirect control of intersection angle at interfaces, so that line slopes can be made approximately continuous.

In Ref. 193 finer grids are embedded into an overall coarser grid by simply adding points between lines of the coarse grid in certain regions. Another approach to complicated configurations is to overlay coordinate systems of different types, or those generated for different subregions, as in Ref. 12. Here an appropriate grid is generated to fit each individual component of the configuration, such that each grid has several lines of overlap with an adjacent grid. Interpolation is then used in the region of overlap when solutions are done on the composite grid, with iteration among the various grids. Transonic potential solutions have been reproduced successfully with different patterns of overlap. This approach has the advantage of simplicity, in that the various subregion grids are only required to overlap, not to fit. However, there would appear to be problems if regions of strong gradients fall on the overlap regions. Also, the interpolation may have to be constructed differently for different configurations, so that a general code may be hard to produce. More recent applications of overlapping grids have been given in Refs. 175 and 15, and good results for Euler solutions for transonic flow over multiple airfoils have been obtained. Simple overlapping of grids was used also in Ref. 102.

In some configurations of the computational field, grid points can occur which have more than the usual number of coordinate lines intersecting at a point. Such singular points generally require special treatment as noted in Ref. 184. Such points were handled in Ref. 115 and also in Ref. 82 without serious error, although some effects of skewness were seen. It is generally easier to treat such points in finite volume formulations or other integral conservation forms. In Ref. 106 the special points that occur where three subregion grids intersect are treated by allowing the grids to overlap and then interpolating as needed. The treatment of such specified points has recently been discussed in Refs. 54 and 86.

In Ref. 34 such singularities can be placed inside cells, rather that at grid points. *B*-spline mapping is used in Ref. 151 to model through singularities inside cells. In Ref. 139 the axis singularity in a rotated axisymmetric system is located inside the cells. This necessitates use of an averaged value on the axis, and this value was found to be in some error.

Order of Difference Expressions

Difference representations on curvilinear coordinate systems are constructed by first transforming derivatives with respect to Cartesian coordinates into expressions involving derivatives with respect to the curvilinear coordinate and derivatives of the Cartesian coordinates with respect to the curvilinear (metric) coefficients. The derivatives with respect to the curvilinear coordinates are then replaced with difference expressions on the uniform grid in the transformed region.

The order of difference expression on curvilinear coordinate systems has been considered in some detail in Ref. 188. The order of a difference representation refers to the exponential rate of decrease of the truncation error with the point spacing. On a uniform grid this concerns simply the behavior of the error with a decrease in the point spacing. With a nonuniform point distribution there is some ambiguity in the interpretation of order in that the minimum spacing may be decreased either by increasing the number of points in the field or by changing the distribution of a fixed number of points. Of course, both of these could be done simultaneously, or the points could even be moved randomly, but to be meaningful the order of a difference representation must relate to the error behavior as the point spacing is decreased according to some pattern. This is a moot point with uniform spacing, but two senses of order on a nonuniform grid emerge: the behavior of the error as 1) the number of points in the field is increased while maintaining the same relative point distribution over the field, or 2) the point distribution over the field is changed so as to reduce the minimum spacing with a fixed number of points in the field. Other studies of error on curvilinear coordinate systems have been reported in Refs. 83, 198, 112, and 113.

In Ref. 188 it is shown that all difference representations maintain their order on a nonuniform grid with any distribution of points, in the formal sense of the truncation error decreasing as the number of points is increased while maintaining the same relative point distribution over the field. The critical point here is that the same relative point distribution is maintained as the number of points in the field is increased. If this is the case, then the error will be decreased by a factor that is a power of the inverse of the number of points in the field as this number is increased. However, random additions of points will not maintain order. This point has also been noted by Hoffman in Ref. 83. In a practical vein this means that a solution made with twice the number of points as another solution will exhibit one-fourth of the error (for second-order representations in the transformed plane) when the two solutions use the same relative point distribution. However, if the number of points is doubled without maintaining the same relative distribution, the error reduction will not be as great as one-fourth.

From the standpoint of formal order in this sense, then, there is no need for concern over the form of the point distribution. However, formal order in this sense relates only to the behavior of the truncation error as the number of points is increased, and the coefficients in the series for the error may become large as the parameters in the distribution are altered to reduce the minimum spacing with a given number of points in the field. Thus, although the error will be reduced by the same order for all point distributions as the number of points is increased, certain distributions will have smaller error than others with a given number of points in the field, since the coefficients in the series, while independent of the number of points, are dependent on the distribution.

The nonuniform grid does introduce a diffusion-like truncation error term which is proportional to the product of the rate of change of the grid spacing and the second derivative of the solution. Since this numerical diffusion may be negative, this effect is potentially destabilizing. Therefore the grid spacing should not be allowed to change too rapidly.

A sufficient condition for the maintainence of order on a nonuniform grid when the minimum spacing is decreased, while keeping the same total number of points, is that the q derivative of x be proportional to the q power of x_ξ. Where this is the case, the order of the difference representation is maintained with the nonuniform point distribution in the sense that the truncation error is reduced by a factor equal to a power of the spacing as the spacing is decreased with a fixed number of points in the field. This, however, is a very severe restriction that is not likely to be achieved in practice.

Several point distributions were examined in Refs. 188 and 198 in regard to truncation error, and the following conclusions were reached:

1) The exponential is not as good as the hyperbolic tangent and therefore should not be used.

2) The hyperbolic sine is the best function in the lower part of a boundary layer. Otherwise this function is not as good as the hyperbolic tangent.

3) The error function and the hyperbolic tangent are the best functions outside a boundary layer. Between these two, the hyperbolic tangent is the better within the boundary layer, while the error function is the better outside.

4) The logarithm, sine, tangent, arctangent, inverse hyperbolic tangent, quadratic, and also the inverse hyperbolic sine are not suitable. The hyperbolic tangent distribution functions of Ref. 198 have been used recently in Refs. 31, 40, and 138.

It is also shown in Ref. 188 that the use of numerical evaluation of the metric coefficients, rather than exact analytical evaluation, eliminates a term from the truncation error. Since this particular term is the most troublesome part of the error, being dependent on the same derivative being represented, it is clear that numerical evaluation of the metric coefficients *by the same difference representation used for the function* whose derivative is being represented is preferable to exact analytical evaluation. It should be understood that there is no incentive, per se, for accuracy in the metric coefficients, since the object is simply to represent a discrete solution accurately, not to represent the solution on some particular coordinate system. The only reason for using any function at all to define the point distribution is to ensure a smooth distribution. There is no reason that the representations of the coordinate derivatives have to be accurate representations of the analytical derivatives of that particular distribution function.

Finally, it is shown that reasonable departure from orthogonality is of little concern when the rate of change of grid spacing is reasonable. Large departure from orthogonality may be more of a problem at boundaries, where one-sided difference expressions are needed. Therefore, grids should probably be made as nearly orthogonal at the boundaries as is practical.

Conservative Forms

When the partial differential equations to be solved on the grid are differenced in conservative form, it is possible for the metric coefficients to introduce spurious source terms into the equations, as has been noted in several works cited in Ref. 190 and discussed also in Refs. 184, 174, and 81. This occurs because in the conservative form the metric coefficients are brought inside the difference operators, and if the differencing of these coefficients does not result in exact numerical satisfaction of the metric identities, then nonvanishing terms will remain in the expressions of the gradient of uniform physical quantities. These metric identities are obtained from the differential equations when the dependent variables are all uniform.

This effect is illustrated simply by consideration of the following conservative and nonconservative forms of a first derivative:

$$Jf_x = (fy_\eta)_\xi - (fy_\xi)_\eta = f_\xi y_\eta - f_\eta y_\xi \tag{1}$$

If f is uniform, the nonconservative form clearly gives a vanishing f_x. However, this is not the case with the conservative form unless the differencing is such that $(y_\eta)_\xi = (y_\xi)_\eta$ numerically. In particular, if the metric coefficients are evaluated analytically, this identity will not be satisfied numerically when these coefficients are differenced. (This is true even in the simple case of cylindrical coordinates.) This illustrates the important fact, also alluded to above, that it is not how accurately the metric coefficients are evaluated that is important, but how accurate are the overall difference expressions.

This effect extends also to metric identities between space and time differences when the grid is time-dependent. Here the conservative difference form of the continuity equation will reduce to a metric identity which involves the time derivative of the Jacobian when the dependent variable is uniform. If this identity is not satisfied exactly, this equation becomes an evolution equation for the Jacobian. Thus, it may be necessary to evaluate the Jacobian from this equation rather than directly from the coordinate derivatives for use in some places in the equation, while the direct evaluation is used in others. Several relevant references are cited in Ref. 190, one of which has also appeared later as Ref. 81.

It is possible in many cases to achieve exact numerical satisfaction of the metric identities through careful attention to the differencing and the evolution of the metric coefficients. As noted above, these coefficients should be expressed by differences, not analytically. The metric coefficients should be evaluated directly from coordinate values wherever they are needed. The metric coefficients should never be averaged, since use of averaged values will almost certainly result in lack of satisfaction of the metric identities. Values of the coordinates at points between the grid points that are needed to construct difference expressions that will satisfy the metric identities can be obtained by averaging between the grid points. Another alternative is to generate a coordinate grid with twice as many points in each direction as are to be used in the physical solution. In Ref. 157 this direct evaluation of the metric coefficients at all points needed in the difference expressions did, in fact, eliminate problems with the metric identities. Further discussion of the construction of different formulations so as to satisfy the metric identities has been given recently in Refs. 105 and 54.

The exact satisfaction of the identities becomes more difficult in three dimensions, and in schemes involving higher order operators or unsymmetric difference expressions. When exact satisfaction is not achieved, the effects of the spurious source terms can be partially corrected, as discussed in Ref. 174, by subtracting off the product of the metric identities with either a uniform solution or the local solution. The former amounts to using a kind of perturbation form, while the latter is, in effect, expansion of the product derivatives involving the metric coefficients and retention of the supposedly vanishing terms, thus putting the equations into a weak conservation law form. Subtraction of the product with the uniform freestream solution was used in Refs. 19, 162, and 160 because of the difficulty in satisfying the metric identities exactly with flux-vector splitting which involves directional differences.

Error Evaluation and Reduction

A clear example of the need for a nonuniform grid in regions of strong solution variation is provided in Ref. 39. Here both grids are for the same boundary, having a corner where an expansion occurs in compressible flow. One grid has uniform spacing along the boundary, while the other is clustered at the expansion corner. On the plot of boundary pressure, the circles are experimental data for the boundary used in the grid. The triangles, however, are for the same boundary but with a rounded corner. Note that while the results from the clustered grid agree well with the data of the actual corner, those for the uniform grid are closer to the data for the rounded corner. The uniform grid at the sharp corner thus has the effect of artificially rounding the corner.

The derivation of expressions for the truncation error in two and three dimensions is straightforward, following the procedures described above, i.e., 1) expression of the Cartesian derivative in terms of the curvilinear derivatives, 2) derivation of the truncation error of the difference expression for the curvilinear derivatives by Taylor series on the uniform computational grid with unit spacing, 3) transformation of all the curvilinear derivatives, and 4) substitution into the expression for the Cartesian derivative being represented. These derivations can be quite tedious, and the use of symbolic manipulation codes such as MACSYMA, noted in Ref. 150, may lead to more efficient and error-free derivatives in general.

The criteria for point distribution functions that will allow the order of difference representations to be retained on a uniform grid discussed above are based on selecting distribution functions for which the higher derivatives are related to the first derivative (the spacing) such that all terms of the truncation error vanish with the spacing. The only dependence on the solution function in these criteria was the limitation on the spacing imposed in Ref. 198.

Another approach is possible when some reasonable estimate of the solution function is available, or in the adaptive grid systems discussed later. If the grid points are located such that the solution changes by the same amount between each pair of points, then the second and higher derivatives of the solution function with respect to the curvilinear coordinate will all vanish, and, consequently, the difference expression will have truncation error due only to that involved in the evaluation of the metric coefficients. (Note that this is a situation in which analytical metric evaluation would yield higher overall accuracy than numerical evaluation.) This, however, is somewhat of an idealistic approach, since it presupposes knowledge of the solution function and, further, would be difficult to achieve in higher dimensions. It does though, give some guidance as to the placement of the points. The type of distribution that this approach produces should still be examined in the light of the above analysis of distribution functions in regard to order, since, as pointed out in Ref. 198, if the distribution can be inappropriate in this respect, the accuracy may be seriously degraded when the numerical solution differs from that used to design the distribution.

In boundary-layer applications it is reasonable to form such a point distribution from a velocity profile across the layer, placing the points such that the velocity profile across the layer, placing the points such that the velocity changes by the same amount between each pair. However, for turbulent boundary layers, a composite function is usually necessary to describe the profile, and such a function has been used in Ref. 21. This distribution, in fact, is used in a procedure in which the next line of points is generated at each successive station of a marching boundary-layer solution, based on the solution at the present station. Another example of the use of a point distribution function designed in regard to a boundary layer is given in Ref. 78. The development of such functions is discussed in more detail in Ref. 190.

Reference 98 gives some direction toward separating the various influences of the grid on the truncation error as expressed in terms of the metric coefficients. However, for a clear statement of the error to be obtained, the derivatives with respect to the curvilinear coordinates should be expressed in terms of the derivatives with respect to the Cartesian coordinates, and the increments in the curvilinear coordinates should be eliminated, as noted above.

Further comments on the influence of the grid qualities on the accuracy of numerical solutions done thereon are given in Ref. 174, where it is noted that the use of compact differencing seems to reduce the deleterious effects of the rate of change of the grid spacing.

Grid Generation Systems

Grid generation systems are procedures for generating the curvilinear coordinate system which defines the grid. Reference 184 discusses the general ideas involved, and the various approaches are surveyed in Ref. 190. These systems fall into two basic classes: algebraic systems, in which the coordinates are determined by interpolation, and partial differential equation systems, in which the coordinates are the solution of these equations. All of the following subsections, except the first, deal with procedures that are of this latter class.

Algebraic Systems

Algebraic grid generation is the fastest procedure in many cases, and its use is surveyed in Refs. 168, 190, and 169. Algebraic procedures also allow explicit control of the grid point distribution. As has been discussed in Ref. 190, some algebraic grid generation systems propagate boundary slope discontinuities into the field, and there is no inherent smoothing mechanism. Some problems in this regard were experienced in Ref. 25. However, the use of local interpolation in the multisurface method can prevent this propagation of discontinuities into the field (cf., Ref. 50).

The algebraic approach is particularly attractive for use with interactive graphics since grids can be produced quickly. Such an interactive approach is discussed in Ref. 170. Here curves are digitized with the cursor, fitted by smoothed cubic splines, analyzed for quality, and revised until the metric coefficients are satisfactorily bounded. The complete grid is then generated by the "two-boundary" form[168] of transfinite interpolation discussed below.

Algebraic grid generation is basically an interpolation among boundaries and/or intermediate surfaces in the field. Simple one-dimensional stretching involves only the use of transformation functions and is often applied to a coordinate system generated by other means, as noted elsewhere in this review. Reference 190 cites a number of such applications. Recent examples of simple stretching along straight lines are given in Refs. 197, 24, 203, 40, 111, and 132. Another simple application is the normalization of the separation between two boundaries as has often been used (cf., Refs. 124, 149, and 191) for recent applications.

In the more general case, transfinite interpolation (discussed in some detail in Ref. 66) is among curves or surfaces. Transfinite interpolation, originally developed for computer-aided design of sculptured surfaces and solids, involves interpolation among functions defined along curves or surfaces, rather than among point values, and thus matches the function at a nondenumerable number of points. [The nondenumerable aspect of transfinite interpolation comes from the possible infinity of points defining general boundaries as compared to a tensor product structure (i.e., product of projectors) that uses only corner information and, therefore, is finite.] In actual application, however, these functions may be defined by discrete sets of values, i.e., piecewise linear functions. In higher dimensions this interpolation can be stated as a sequence of univariate interpolations, i.e., projections, which are put together as Boolean sum projections. (The Boolean sum is the primary mechanism for defining transfinite interpolation. The multidirectional results are Boolean sums of unidirectional interpolations.) The functions specify the values (and perhaps some derivatives) of the variables on the curves or surfaces. Values in the interior between these curves or surfaces are determined by interpolation, using specified interpolation functions usually called blending functions. The blending functions are often polynomials, but other functions can be used also. Transfinite interpolation is used to generate grids joined to an analytically generated grid near a corner in Ref. 199. A recent application to two-dimensional airfoils appears in Ref. 193.

The various interpolation methods differ primarily in regard to how many and what curves or surfaces are used, and what derivatives, if any, are specified on these surfaces, and secondarily in regard to what type of blending functions are used. Several approaches are discussed in Refs. 190, 168, and 66. The order of accuracy of the interpolation may be increased either by adding more curves or surfaces, or by adding more information, e.g., specification of higher derivatives on the curves or surfaces. Transfinite interpolation is used in Ref. 53 for three-dimensional grid generation, and Refs. 163 and 68 give recent applications.

Intermediate surfaces within the region may be necessary with severely distorted regions for which interpolation only between boundaries would result in unsatisfactory grids. Reference 66 shows an example where the use of just an intermediate point in the field corrected a coordinate system that had overlapped the boundary. The "two-boundary" technique (used recently in Refs. 101 and 159) and the "outer surface" method (both discussed in Ref. 168) are examples of one-dimensional Hermite interpolation using only the opposing boundary surfaces. In Ref. 110 the overall grid consists of subregions, in each of which the grid is generated by boundary interpolation using certain discrete boundary points, as is common in finite element analysis.

The multisurface method, discussed also in Refs. 168 and 190, is a related unidirectional interpolation procedure. This procedure is constructed from an interpolation of a specified vector field followed by vector normalizations at each interpolation point in order to cause a desired telescopic collapse so that the boundaries are matched. The specified vector field is defined from piecewise linear curves determined by the boundaries and successive intermediate control surfaces. Normals to such surfaces are special cases. Polynomial interpolants for the vector field yield all of the classical polynomial cases in physical space along with a rational method for avoiding disasters such as direct Hermite interpolation with excessively large or discontinuous derivatives. Here the immediate surfaces are not coordinate surfaces, but are used only to define the normal vector field. These vectors are taken to be tangents to the coordinate lines intersecting the surfaces so that integration of this vector field produces the position vector field for the grid points. In Ref. 49 the multisurface method is applied to generate embedded grids, as discussed above, with continuity of coordinate line slope, using local piecewise-linear interpolants. This approach is extended in Ref. 50 to use higher order local interpolants to allow for curvature continuity as well. This procedure contains both linear and Hermite interpolation as special cases, and, as noted in Refs. 47 and 48, it can be brought into the transfinite context by means of Boolean sums.

A collection of subroutines which automatically perform the necessary parts of grid construction using this multisurface procedure has been written and is described in Ref. 47. Some of the automation features of this collection are applicable to other grid construction procedures as well. These subroutines can rotate and move curves, project one curve from another, normalize and reparameterize curves, cluster points on a curve, and perform other such utilitarian functions to aid in the setup of an overall configuration. The multisurface method has recently been applied in Refs. 4, 31, and 138 in transonic flow solutions.

In a manner similar to that referred to earlier for a stretching transformation of a grid generated by some other means, stretching functions can be embedded in the blending functions of the interpolation methods to allow control of the point spacing in all of these grids generated by interpolation.

Conformal Mapping

Innovations in conformal mapping continue to extend this classical technique to more complicated configurations, and surveys of the various techniques available are given in Refs. 88 and 190. Some specific recommendations of techniques and tools are given in Ref. 88 and the advantages of conformal mapping are stated succinctly in Ref. 207. Conformal systems have the advantage of introducing the fewest additional terms in transformed partial differential equations. Considerable understanding of the theory of functions of a complex variable may be necessary for effective applications though.[207] In Ref. 129 several simple functions that will generate symmetric duct-like configurations with deformed walls have recently been given. No provision is made for fitting an arbitrary duct, however. A duct was also treated in Ref. 133.

Although the complex variable techniques by which conformal transformations are usually generated are inherently two-dimensional, certain more general cases can be treated by rotating or stacking two-dimensional systems, cf., Ref. 36. Reference 88 gives an example of an axisymmetric system produced by rotating a planar system about a central axis. This procedure is used for cascades in Refs. 42, 95, and 96. Systems can also be generated on curved surfaces, as has been done by cartographers. Grids generated following Ref. 42 have been used recently in Ref. 126. Sequences of conformal transformations and shearing transformations have been used recently in Ref. 165 to generate stacked two-dimensional grids for wing-tail-fuselage combinations. Such sequences have also been used in Refs. 97, 91, and 178. In Ref. 91 the procedure is constructed so as to allow easy addition of components. Further examples of the use of conformal mapping in the construction of three-dimensional configurations are given in Refs. 89, 70, 171.

The trend in treating more complicated regions is to break the mapping up into a sequence of more simple mappings. Contours such as airfoils are generally mapped to near-circles by one or more simple transformations, and then the near-circle is mapped to a circle by a series transformation, e.g., the Theodorsen procedure. It is necessary for convergence that the near-circle be sufficiently near to being a circle. Reference 5 notes that the efficiency of the Theodorsen transformation is dependent on the closeness of the near-circle to a circle, and a procedure is given for selecting the parameters in this first transformation. A sequence of transformations is used in Refs. 94, 120, and 90, the latter usage being in a design procedure.

In Ref. 74, a series for the differential form is shown to be superior to the usual Theodorsen form for general bodies. This series appears in terms of arc length and surface angle, rather than the polar coordinates of the Theodorsen form which can lead to infinite derivatives and multiple values. The ordering of the points can break down in the Theodorsen form for closely spaced points also. The differential form is applicable, however, as long as there are no corners, even for twisted contours. In this and other series transformations, the differential form is usually more tolerant of odd shapes. The differential form has been used recently in Ref. 36.

Recent extensions of the Schwarz-Christoffel transformation to curved contours have made this procedure a powerful tool for treating complicated internal and other configurations. These improvements, discussed in Refs. 190 and 88, also lead to smoother metric coefficients for boundaries with slope discontinuities than in older methods for the Schwarz-Christoffel transformation, as noted in Ref. 10 where application is made to turbine passages. This procedure was also used in Refs. 62 and 177. This procedure for the Schwarz-Christoffel transformation also may be more efficient than other conformal procedures involving an intermediate mapping of a near-circle for mapping contours and circles in some cases.[88] Extensions to exterior free streamline flows have been given recently in Ref. 79. A slightly different approach to the Schwarz-Christoffel transformation is given in Ref. 207. The Schwarz-Christoffel transformation in a different form has also been used recently in Ref. 22. Two successive Schwarz-Christoffel transformations are used in Ref. 195 for straight-sided enclosures.

Multiple-body configurations can be treated by a sequence of transformations which map each body to a circle in succession while maintaining previously established circles. Another procedure, discussed in Ref. 73, involves iteratively mapping each body to a circle with no special consideration of the others. This process generally requires only a few iterations to converge. A third approach, also discussed in Ref. 73, involves connecting all of the bodies in a string and mapping the resulting (effective) single body. This procedure is the simplest, but will not give satisfactory grids for closely spaced bodies in general. Different grids can be constructed from the streamlines and potential lines for the multiple-body configurations by assigning selected values of circulation at each body, as in Ref. 73. This produces slit-type (H-type) systems, which can have bad spacing near stagnation points of the streamlines when circulation solutions about the bodies are combined with a freestream. Without the freestream, O-type grids are obtained with nonzero circulation about one body and zero circulation about the others. Again there is large spacing near the "stagnation" points. These stagnation points can be removed by alternating the sign of the circulation from one body to the next and adjusting the magnitudes so that the total circulation vanishes, but now there is large spacing far from the bodies. This situation can be improved, however, by adding an outer contour surrounding all bodies. One final approach involves the segmentation of the region into subregions and is discussed elsewhere in this review. Also noted elsewhere is the approach of Ref. 76 where the bodies are all mapped to rectangles.

Conformal mappings may also be constructed numerically, and several approaches are noted in Ref. 190. An integral method based on the formulation of Symm (cf., Ref. 190) has been given recently in Ref. 85 for both interior and exterior regions. Other recent numerical procedures are given in Refs. 17, 71 and 134. The use of the complex variable techniques with a sequence of simple transformations, however, generally has been preferred. It is sometimes more efficient to generate the final grid by solving the Laplace system numerically with Dirichlet boundary conditions from the conformal transformations, especially if a fast Poisson solver can be applied. One numerical approach that has had some use is the construction of streamlines and potential lines of potential flow by superposition of singularities. This approach is noted in Ref. 190 and also is used in Ref. 100 and in Ref. 73 for multiple bodies. This approach generally results in slit-type configurations unless a zero freestream is used with circulation on the bodies, as in this latter reference.

Control of coordinate line spacing is a problem with conformal systems, but, as noted elsewhere in this review, the application of one-dimensional stretching transformations to a conformal system will leave the system orthogonal, although not conformal. An example appears in Ref. 62. Even if more general stretching transformations are applied, the system may not depart greatly from orthogonality. Reference 42 makes extensive use of this approach, and other examples are in Refs. 28, 30, 95, 119, and 38. Shearing transformations are also often used in conjunction with conformal transformations to map boundaries onto straight lines, etc. The resulting system is no longer conformal, of course. Recent examples appear in Refs. 165, 97, 91, 33, and 70. In Ref. 41 a sequence of Schwarz-Christoffel and shearing transformations is applied to generate an O-type nearly orthogonal system for a two-dimensional airfoil in a wind tunnel in which the upstream and downstream infinites map to points.

Elliptic Systems

Elliptic partial differential systems have certain advantages as generation systems for coordinates, as discussed in Refs. 185 and 190. This first reference also discusses the properties, solutions, and applications of such generation systems in an introductory manner. The effects of the control functions used in these systems to control coordinate line distributions and orientations are also discussed, as is, in particular, the treatment of these functions in the presence of branch cuts onto different Riemann sheets.

Various different types of equations have been considered, as discussed in Ref. 190, but the best general choice seems to be those based on some consideration of differential geometry, cf., Ref. 200. Linearization by replacement of certain metric coefficients in these equations with specified functions can be considered, as noted in Ref. 190, but this approach is likely to lead to distorted grids with more general shapes and configurations.

The most widely used elliptic generation system is that based on the system of Poisson-like equations

$$\nabla^2 \xi^i = P^i \qquad i=1,2,3 \tag{2}$$

where the functions P^i serve to control the coordinate line spacing. The transformed equations are, in vector form,

$$g^{ij} r_{\xi^i \xi^j} + P^i r_{\xi^i} = 0 \tag{3}$$

This system has been used in many works as noted in Ref. 190 and discussed also in Ref. 185. Recent uses are given in Refs. 142, 46, 13, 20, 196, and 130. In Refs. 20 and 196 the transformation was from cylindrical, rather than Cartesian, coordinates.

It is shown in Ref. 200 that if a coordinate system, ξ^i, generated from the homogeneous equation, i.e., with $P^i = 0$, is transformed to another system, ξ^i, then the new system is the solution of Eq. (2) with the control functions

$$P^k = g^{ij} P^k_{ij} \qquad k=1,2,3 \tag{4}$$

where

$$P^k_{ij} = \sum_{m=1}^{3} \sum_{n=1}^{3} \frac{\partial \xi^m}{\partial \xi^i} \frac{\partial \xi^n}{\partial \xi^j} \frac{\partial^2 \xi^k}{\partial \xi^m \partial \xi^n} \tag{5}$$

These results show that a coordinate system generated by applying a stretching transformation to a system generated from the homogeneous equations could have been generated directly from the inhomogeneous equations with the proper control functions. (In some cases it may be preferable, as noted in Ref. 127, to generate the grid by applying stretching to a preliminary grid generated from Laplace equations, since the convergence of the equivalent elliptic system becomes more difficult as the control functions become stronger. Grids generated using this procedure have been used recently in Refs. 72, 111, and 160.) Of particular interest is the structure of the control functions for both static and dynamically adaptive use. The form given by Eq. (4) is more general than the form given in Ref. 185 in the inclusion of the terms involving the off-diagonal metric coefficients as well as in the control functions. These terms correspond to stretchings that are not one-dimensional. For one-dimensional stretching, $\xi^k = f_i(\xi^k)$, only P^1_{11}, P^2_{22}, and P^3_{33} are nonvanishing, so that

$$P^k = g^{kk} P^k_{kk} = f''_k / f'^2_k \qquad \text{(no summation)} \tag{6}$$

This result is of particular importance in that it shows that the more general form given by Eq. (4) for the control functions therefore should be used, rather that that given in Ref. 185 [see Eq. (8) in that reference], with the P^k_{ij} then considered to be the control functions to be specified.

Several investigators have been led to the special case of this form given in Ref. 185, as is noted in that reference and in Ref. 190, because of the simplicity of its one-dimensional form which allows one-dimensional control functions to be determined by quadrature from a specified point distribution, i.e., by integration of Eq. (6). References 164, 179, and 180 all use this form. This form also results in control functions that are smaller than those in Eq. (2) by several orders of magnitude.

Equation (5) provides the precise form of these control functions corresponding to application of multidimensional stretching to a solution of the homogeneous equations, but these functions are free to be constructed in any manner, e.g., as sums of exponentials as discussed in Refs. 185 and 164. In this latter reference the decay factor of an exponential is adjusted iteratively until a desired arc spacing along a coordinate line is achieved.

Reference 172 discusses the GRAPE code which generates a two-dimensional grid from an elliptic generation system by iteratively adjusting the control functions until specified point and intersection angle distributions are obtained on the boundaries using the system of Eq. (2). This code has recently been applied in Refs. 109 and 105. This approach has also been used in three dimensions in Refs. 173, 161, and 58. In Ref. 58 the transformation is from polar coordinates, rather than Cartesian coordinates.

It is often advisable to determine the control functions in the interior of a region by interpolation of values found by applying limiting forms of the generation system of reduced dimensionality on the boundaries of the region. Thus control functions on a surface can be determined by interpolation between the edges where the functions are evaluated by substituting the specified boundary point distribution into a one-dimensional form of the generation system (along a curved line in general). A similar two-dimensional evaluation using specified or generated points on a curved boundary surface can provide boundary values of the control functions for interpolation into the three-dimensional field. The development of the limiting lower dimensional forms of the generation system involves taking projections of the equations along curved lines or surfaces, as is discussed in Ref. 180. This approach is discussed also in Refs. 153 and 185 and is used in Refs. 180 and 107 as well. As noted above, this approach can be used to generate coordinate systems in subregions which can be patched together to form a composite grid for general three-dimensional regions.

It should be noted, as in Ref. 185, that boundary point distributions alone are not sufficient to establish the same line distribution in the field, even in simply connected regions. It is necessary to include control functions evaluated from the boundary point distributions in the manner discussed above, or else the inherent smoothing of the Laplacian operator will tend to produce equal spacing in the field regardless of the point spacing on the boundary. This point was not made in Ref. 205.

In Ref. 32, a second-order elliptic system is developed by combining the orthogonality conditions, $g_{ij} = 0$, $i \neq j$, and a specification of the Jacobian over the field, $\sqrt{g} = f(x,y,z)$.

A three-dimensional elliptic generation system that can produce nearly conformal coordinate systems in some cases is constructed in Ref. 27. This three-equation generation system is constructed as a linear combination of the nine equations obtained by writing the basic three-dimensional elliptic system of Eq. (2) three times, in each case dropping the derivatives with respect to one curvilinear coordinate. The off-diagonal metric coefficients are also set to zero, and all control functions are dropped except for one of the diagonal elements. The nonlinear coefficients in these equations are evaluated from a known conformal transformation function on facing coordinate surfaces and interpolated into the field. A C-type system around the wing leading edge, with a fictitious extension extrapolated beyond the wing tip, is obtained for a wing-body configuration. This generation system was used in Ref. 29.

In Ref. 166, the Cartesian coordinates are taken to be solutions of fourth-order biharmonic equations. This system requires two boundary conditions on the entire boundary, so that the intersection angle can also be specified. This system is then linear in the transformed region and nonlinear in the physical region, so that maximum principles are lost and overlap of the coordinate lines is possible with some configurations or boundary specifications. It is generally preferable to have linear operators in the transformed region, rather than the physical region, in order to allow general configurations to be treated, as noted in Ref. 190.

In three dimensions it is possible, as shown in Ref. 200, to devise a generation system based on the property of zero curvature that characterizes a Euclidean space. Setting the Riemann curvature tensor to zero yields six second-order partial differential equations with the six metric coefficients as dependent variables. In two dimensions, five of the six equations are lost, but three metric coefficients remain, so that two additional constraints, e.g., orthogonality conditions, are necessary to close the system. A two-dimensional orthogonal generation system based on these equations is given in Ref. 201.

It is only the metric coefficients, and perhaps their derivatives (the Christoffel symbols), that are needed in the transformed partial differential equations of physical systems. However, the Cartesian coordinates can be obtained from the metric coefficients by solving a system of first-order partial differential equations, as shown in Ref. 200. For two-dimensional orthogonal systems this final solution reduces to quadratures.

In Ref. 77, the equations of an elliptic generation system are solved by checkerboard SOR, an iterative scheme which will vectorize. References 18 and 156 use Jocobi iteration in order to use vectorization. The three-dimensional solution in Ref. 179 uses line SOR, as does the two-dimensional procedure in Ref. 107. Alternating-direction (ADI) solutions are used in Refs. 32, 205, 84, and 58. A semidirect marching method of solution is given in Ref. 150. However, this procedure is not suitable when there is significant grid clustering in more than one curvilinear coordinate direction. A multigrid solution is given in Ref. 20.

Since the elliptic generation systems normally used are nonlinear, the initial guess must be sufficiently near the solution or else iterative solution methods will not converge. Reference 150 suggests gradually building up the nonlinear coefficients as the iteration proceeds, starting with uniform values and hence linear equations which, if convergent, are so from any initial guess. This reference also suggests gradual building up of the boundary conditions, and gradual implementation of the control functions, as noted also in Ref. 190.

The solution of the biharmonic generation system of Ref. 166 was done by an alternation between the conjugate gradient and Gauss-Seidel iteration methods. The former works well in resolving low-frequency error waves, while the latter is more effective against the high-frequency waves.

Finally, an elliptic generation system can be applied as a smoother to a grid generated by any means, as in Refs. 91 and 56.

Parabolic and Hyperbolic Systems

Reference 122 uses parabolic partial differential equations as the generation system. Hyperbolic equations have also been used, as noted in Ref. 190. Hyperbolic systems do not allow the entire boundary to be specified, rather the solution marches outward from a specified boundary, the outer boundary being free. Hyperbolic systems will propagate boundary slope discontinuities into the field. The solution of both of these systems is typically easier than that of elliptic systems, since marching tridiagonal solutions can be used if an appropriate linearization is done.

Reference 11 uses a parabolic system formed by adding a time derivative to the usual elliptic system in order to obtain grid evolution equations to couple with the flow solution equations for a time-dependent free surface. In Refs. 125 and 127 a hyperbolic generation system is used in which the Jacobian, i.e., the cell area, can be specified. The resulting coordinate system is orthogonal at the boundary and nearly so elsewhere. This generation system has been discussed also in Ref. 190.

Orthogonal System

Orthogonal coordinate systems produce fewer additional terms in transformed partial differential equations. Also, as noted earlier, severe departure from orthogonality will introduce truncation error in difference expressions. A general discussion of orthogonal systems on planes and curved surfaces is given in Ref. 48 and various generation procedures are surveyed in Refs. 48 and 190. For orthogonal systems all of the off-diagonal metric coefficients vanish, and for conformal systems the remaining coefficients are all equal.

In numerical solutions, the concept of numerical orthogonality, i.e., that the off-diagonal metric coefficients vanish when evaluated numerically, is usually more important than strict analytical orthogonality, as noted in Ref. 48, especially when the equations to be solved on the system are in the conservation law form.

There are basically two types of orthogonal generation systems: one based on the construction of an orthogonal system from a nonorthogonal system, and the other involving field solutions of partial differential equations. The first approach involves the construction of orthogonal trajectories on a given nonorthogonal system. Here one set of coordinate lines of the nonorthogonal system is retained, while the other set is replaced by lines emanating from a boundary and constructed by integration across the field so as to cross each line of the retained set orthogonally. Control of the line spacing is exercised through the nonorthogonal system and through the point distribution on the boundary from which the trajectories start. The point distribution on only three of the four boundaries can be specified. Several methods for the construction of orthogonal trajectories are discussed in Refs. 48 and 190 and some recommendations are made.

If point distributions are to be specified on all boundaries, the field approach must be taken. As noted in Refs. 200 and 190 all orthogonal coordinate systems must satisfy the equation

$$\nabla^2 \xi^i = \frac{1}{\sqrt{g}} \frac{\partial}{\partial \xi^i} \left(\frac{h_j h_k}{h_i} \right) \tag{7}$$

where i, j, k are cyclic and no summation is intended. Here $h_i = \sqrt{g_{ii}}$ with no summation. The off-diagonal metric coefficients (g_{ij}, $i \neq j$) vanish for orthogonal systems. The transformed equations are

$$L_0 r = 0 \tag{8}$$

where the second-order differential operator L_0 is defined by

$$L_0 \equiv \frac{\partial}{\partial \xi}\left(\frac{h_2 h_3}{h_1}\frac{\partial}{\partial \xi}\right) + \frac{\partial}{\partial \eta}\left(\frac{h_1 h_3}{h_2}\frac{\partial}{\partial \eta}\right) + \frac{\partial}{\partial \zeta}\left(\frac{h_1 h_2}{h_3}\frac{\partial}{\partial \zeta}\right) \tag{9}$$

In two dimensions, the coordinate system satisfies Laplace equations when $g_{22}/g_{11} = \text{const}$ and the conformal case occurs when the constant is unity. It is shown in Refs. 200 and 190 that if a conformal system $(\bar{\xi}, \bar{\eta})$ is transformed to another system (ξ, η) by one-dimensional stretching functions $\xi = f(\bar{\xi})$, $\eta = s(\bar{\eta})$, the new system will remain orthogonal, but not conformal, and will have

$$g_{22}/g_{11} = (f'/s')^2 \tag{10}$$

In Ref. 154, Eq. (9) is applied in two dimensions as the generation system, with g_{22}/g_{11} taken as a (control) function of ξ and η. With this generation system an orthogonal coordinate system can be obtained with arbitrary point distribution on the boundaries by evaluating the single control function in the course of the iterative solution of the generation equation. This is done in a manner similar to that used in the GRAPE code, discussed above, with new boundary values of the control function being calculated from the present iterate for the coordinates. The control function in the field is then determined from these boundary values by either transfinite interpolation or as the solution of Laplace's equations, the former being found preferable in the cases considered. (With more distorted boundaries the Laplace solution might be more reliable than the interpolation.) Different forms of interpolation, or an equation other than the Laplace, for the determination of the control function in the field, would allow some control of coordinate line spacing in the field. However, since only a single control function is involved, it is not possible to exercise control of the coordinate line spacing in the field in both directions.

If one boundary can be left completely free, then a hyperbolic system of equations can be used, with the solution marching outward toward the free boundary. The hyperbolic system can be developed from the requirement that the off-diagonal metric coefficients vanish, with one additional condition on the metric. This additional condition may conveniently be taken as specification of the Jacobian, and hence the cell area, although the cell aspect ratio would be another possible choice. This system has been mentioned in the preceding subsection. Again, several methods are discussed in Refs. 190 and 48. A variational procedure for the generation of orthogonal grids has been given recently in Ref. 121. A procedure based on the vanishing of the Riemann curvature tensor has been given recently in Ref. 35. (The use of the Riemann tensor is also discussed in Ref. 200.)

Orthogonality in three dimensions is difficult to achieve, as noted in Ref. 48, and only exists when the coordinate lines on the bounding surfaces follow lines of curvature, i.e., lines in the direction of maximum or minimum curvature of the surface. Therefore, three-dimensional orthogonal coordinates will not be available in most cases with nontrivial geometry. It is possible, however, to have the system locally orthogonal at boundaries, and/or to have orthogonality of surface coordinates.

Three-Dimensional Systems

The grid generation systems discussed in the preceding subsections, except for the conformal systems, are applicable to general three-dimensional configurations. Several examples of such three-dimensional grids are cited in Ref. 190, and others appear in Refs. 103, 179, 200, and 202. However, with complicated three-dimensional physical configurations, it may be difficult to generate a single grid that is smooth and has adequate point distributions in all areas. One approach, discussed above, is to generate individual systems in contiguous subregions. Single-region grids have been generated, however, in Ref. 163.

Three-dimensional grids may be constructed in some cases by simply connecting corresponding points on two-dimensional grids generated on stacks of planes or curved surfaces. It should be noted, however, that this procedure provides no inherent smoothness in the third direction, except in cases where the stack is formed by an analytical rotation or transformation of the surface system. Stacking of surfaces was used in Refs. 26 and 23 for a three-dimensional internal configuration. Other applications appear in Refs. 25, 125, 197, 142, and 159, where the two-dimensional grids are generated at successive positions in marching solutions. In Ref. 84, only a few two-dimensional systems are generated, and the stack was formed by interpolation between these systems. Another type of stack is constructed by rotating a two-dimensional system, about an axis, as in Refs. 67, 73, 99, 148, 127, and 89. A recent application of stacked grids to treat a wing-tail-fuselage configuration is given in Refs. 165 and 91. Stacking has also been used in Refs. 36, 162, 70, and 160.

Finite wings in such a stacking procedure have been treated as in Refs. 42, 84, 95, and 96 by fictitious extension of the wing with zero thickness beyond its tip. In this approach, the grid points on the extension, which are actually field points, are specified as boundary points. This means that the grid on the extension does not have the benefit of the smoothing effect of the generation system that occurs at other field points, and the metric coefficients may not be continued across the extension. In Ref. 180 the wing is transformed to a slit (H-type), with a fully three-dimensional solution of an elliptic generation system. A full solution is used in Ref. 179 but with a C-type system around the wing tip and its extension forward and aft. This extension of the tip creates an axis singularity which requires special consideration in the solutions done on the grid. Another type of stacking is used in Refs. 139 and 20 where an axisymmetric system is constructed from a succession of algebraically generated systems on coaxial conical surfaces. This approach could be generalized to nonaxisymmetric cases by deforming the grids on the conical surfaces.

In Ref. 56, a pylon/nacelle was inserted into a stacked grid about a wing by simply moving points in the spanwise direction along existing grid lines to the intersection of these grid lines with the pylon/nacelle. The points between the inserted body and some fixed points, some spanwise distance to either side, were then moved proportionally. The resulting grid, however, does not give a really good fit to the entire inserted body. In Ref. 135 three-dimensional systems are generated by locating the intersection of streamlines with planes normal to a fixed direction. These intersections, and thus the coordinate system, are generated in the course of a flow solution. Obviously recirculation cannot be treated; therefore, the range application is somewhat limited.

The generation of a two-dimensional coordinate system on a curved surface is a technique which can provide a surface boundary point distribution for three-dimensional grid generation in a region bounded by surface segments. Such two-dimensional curved surfaces could also be stacked to build up a three-dimensional grid directly. It may be that the generation of two-dimensional grids on curved bounding surfaces is the most difficult part of three-dimensional grid generation, cf., Refs. 153 and 166. Two of the more simple approaches are parametric transformation in which the Cartesian coordinates of the surface are defined in terms of two parametric variables which then become the surface coordinates and projection, whereby a two-dimensional plane system is projected onto the surface. An algebraic construction of a surface grid from a surface parameterization has recently been given in Ref. 163. Projection has been used recently in Ref. 161, and results for transonic potential flow are given in Ref. 162. This approach is applicable only to simple surfaces.

In Ref. 200, a surface generation system based on the Gauss equations of a surface is given. The equations can be put into the form

$$Lr + G_3\left[(\Delta_2\xi)r_\xi + (\Delta_2\eta)r_\eta\right] = \boldsymbol{n}R \tag{11}$$

where Δ_2 is the second-order differential operation of Beltrami, L the second-order differential operator

$$L = g_{22}\frac{\partial^2}{\partial\xi^2} - 2g_{12}\frac{\partial^2}{\partial\xi\partial\eta} + g_{11}\frac{\partial^2}{\partial\eta^2} \tag{12}$$

G_3 is the square of the two-dimensional Jacobian, $G_3 = g_{11}g_{22} - (g_{12}^2)$, $\boldsymbol{n}$ is the normal to the surface, and R is proportional to the mean curvature of the surface. If the generation system for the curvilinear coordinate system on the surface is taken to be

$$\Delta_2 \xi = P, \qquad \Delta_2 \eta = Q \tag{13}$$

where P and Q are control functions to be specified, the transformed equations are, by Eq. (11),

$$Lr + G_3 (Pr_\xi + Qr_\eta) = nR \tag{14}$$

For a plane surface, the Beltrami operator reduces to the Laplacian operator, and the curvature term on the right-hand side of Eq. (14) vanishes, so that this generation system then reduces to the widely used plane two-dimensional system formed by the equations (cf., Ref. 185),

$$\Delta^2 \xi = P, \qquad \Delta^2 \eta = Q \tag{15}$$

which is the two-dimensional form of Eq. (2). The general surface generation system based on the Beltrami operator is thus a generalization of the plane system based on the Laplacian operator. If the control functions are zero (and the boundary points are allowed to move on the boundaries), the resulting coordinates are said to be isothermic. Isothermic coordinates are the general surface analog of conformal coordinates in a plane.

The form given by Eq. (14) is appropriate for a surface of constant ζ, with ξ and η the coordinates on the surface. All of the out-of-surface derivatives, i.e., the ζ derivatives must either be specified externally on the entire surface or interpolated from specified edge values. When this surface generation procedure is used to build a three-dimensional coordinate system by stacking surfaces, the ζ derivatives can be evaluated on specified curves on the specified surfaces which bound the stack and interpolated onto each surface of the stack. The dimensional generation system is then solved on each successive surface in the stack. Note that the surfaces of the stack are not specified, but are generated in turn using the information interpolated from the bounding surfaces. Some exact solutions of Eq. (14) for special cases are given in Ref. 200. The numerical solution procedure is discussed in Refs. 200 and 202, some results being given in the latter work. The required equations are also summarized in Ref. 185.

It is also shown in Ref. 185 that if a solution $(\bar{\xi},\bar{\eta})$ of Eq. (14) with $P = Q = 0$ is transformed to another system (ξ,η), then the new system is the solution of system (14) with the control functions

$$P = (1/G_3)(g_{22}P^1_{11} - 2g_{12}P^1_{12} + g_{11}P^1_{22})$$

$$Q = (1/G_3)(g_{22}P^2_{11} - 2g_{12}P^2_{12} + g_{11}P^2_{22}) \tag{16}$$

where the form of the control function P^k_{ij} is the same as that given by Eq. (5), except that the summation limit is 2. This result shows that a system generated by stretching applied to a solution of the homogeneous equations could have been generated directly from the elliptic system with proper control functions. For one-dimensional stretching, only P^1_{11} and P^2_{22} are nonvanishing, so that this reduces to the simple form given in Eq. (6).

It is also shown in Ref. 200 that the generation system of Ref. 180 for a general surface can be arranged into the same form as Eq. (14). In the derivation of the latter reference, however, the out-of-surface coordinate lines were assumed to intersect the surface orthogonally with vanishing curvature. It was also necessary to divide out a parameter which can be zero in some cases. These steps are not necessary when the development is from the Gauss equations. Finally the interest in the development of the two systems is different, in that in the approach of Ref. 180 the z derivatives are specified. In the Gauss equation procedure of Ref. 200, it is the ζ derivatives that are specified. Both of these approaches are summarized in Ref. 185.

This specification of the out-of-surface derivatives, i.e., the ζ derivatives, can be replaced with the specification of a parametric representation of the surface. In this case the generation system given by Eq. (14) becomes a system of two equations for the two surface parametric variables, rather than the three Cartesian coordinates, as shown in Ref. 189. The surface generation systems of Ref. 180 can also be recast using the parametric surface representation, as has been done in Ref. 206. When cast as a generation system based on parametric representation of the surface, the approach of Ref. 200, i.e., Eq. (14), and that of Ref. 180, lead to exactly the same generation system. A third approach, Ref. 59, has recently been given which is also equivalent to these two with the parametric surface representation.

The generation of three-dimensional grids from elliptic grid generation systems has been surveyed recently in Ref. 189 and algebraic grid generation in three dimensions is discussed in Ref. 169. (The latter reference also discusses computer graphics for use in grid displays.) An algebraic grid generation system based on transfinite interpolation is given in Ref. 53.

Adaptive Grids

Probably the most important area of research in grid generation at present is the development of dynamically adaptive systems in which the grid points move in response to the developing physical solution being done on the grid. The point distribution over the field is thus readjusted dynamically to concentrate points in regions of larger solution variation as they develop, without reliance on prior knowledge of the location of such variations.

Several considerations are involved here, some of which are conflicting. The points must concentrate, and yet no region can be allowed to become devoid of points. The distribution also must retain a sufficient degree of smoothness, and the grid must not become too skewed or the truncation error will be increased as discussed earlier. This means that points must not move independently, but rather each point must somehow be coupled at least to its neighbors. Also, the grid points must not move too far or too fast or oscillations may occur. Finally, the solution error, or other driving influence, must be sensed, and there must be a mechanism for translating this influence into the motion on the grid. The need for a mutual influence among the points calls to mind either some elliptic system, thinking continuously, or some sort of attraction (repulsion) between points, thinking discretely. Both approaches have been taken with some success. The use of an adaptive grid may not necessarily increase the computer time, even though more computations are necessary since convergence properties of the solution may be improved and certainly fewer points will be necessary, cf., Refs. 9 and 45.

In Ref. 194 the boundary point distribution is reparameterized based on the results of a solution on a preliminary grid. Here the reparameterizion is not related either to the error or to the solution variation, but is done through simple functions specified beforehand for certain regions of the flow. This approach is thus not really dynamically adaptive, but rather is a rearrangement of the grid based on one preliminary solution. Nevertheless, the results show considerable improvement in shock resolution.

If the time derivatives are transformed to derivatives taken at fixed points in the transformed region, rather than in the physical region, then no interpolation is required when the adaptive grid moves. This is the appropriate approach when the grid evolves with the solution at each time step. Some methods, however, change the grid only at selected time steps, and here interpolation must be used to transfer the values from the old grid to the new. Illustrations of both approaches are given in Ref. 190 and the latter procedure is used in Ref. 65.

The transformation of the time derivative effectively adds additional convection-like terms, proportional to the grid speed, to the equations. In Ref. 61, the grid is made to move so that the grid speed is such that these additional terms cancel the other first-derivative terms, leaving only second derivatives in the equations to be solved. These first-derivative terms arise both from the convective terms and from the transformation of second derivatives, i.e., from the viscous terms. Without the viscous terms this procedure would amount to using a Lagrangian grid moving with the fluid velocity.

The development and application of adaptive grids has been surveyed in Ref. 187, where it is concluded that the ultimate answer to numerical solution of partial differential equations may well be dynamically adaptive grids, rather than more elaborate difference representations and solution methods. It has been noted by several authors that when the grid is right, most numerical solution methods work well. Oscillations associated with cell Reynolds numbers and shocks in fluid mechanics computations have been shown to be eliminated with adaptive grids. Even the numerical viscosity introduced by upwind differencing is reduced as the grid adapts to regions of large solution variation. The results obtained to date have indicated clearly that accurate numerical solution can be obtained when the grid points are properly located.

It is also clear that there is considerable commonality among the various approaches to adaptive grids. All are essentially variational methods for the extremization of some solution property. The explicit use of variational principles allows effective control to be exercised over the conflicting requirements of smoothness, orthogonality, and concentration, which is probably the most promising approach in multiple dimensions.

The adaptive grid is most effective when it is dynamically coupled with the solution, so that the solution and the grid are solved together in a single continuous problem. The most fruitful directions for future effort thus are probably in the development and direct application of variational principles and in intimate coupling of the grid with the solution.

Variational Approach

Considering the grid from a continuous viewpoint, it occurs that something should be minimized by the grid rearrangement, and thus a variational approach is logical. Such an approach is explained in Ref. 18 and used in Ref. 156. To begin, the quantity $(\nabla\xi)^2+(\nabla\eta)^2$ represents the variation of the curvilinear coordinates over the field in two dimensions, and thus is a measure of the grid roughness. Therefore, to maximize the smoothness of the grid, it is appropriate to minimize the integral of this quantity over the field.

$$I_s \equiv \int [(\nabla\xi)^2+(\nabla\eta)^2]\,\mathrm{d}x\mathrm{d}y \tag{17}$$

The two Euler equations for this variational problem in the physical region are found by setting to zero the application of the following operators to the integrand:

$$\frac{\partial}{\partial\xi}-\frac{\partial}{\partial x}\frac{\partial}{\partial\xi_x}-\frac{\partial}{\partial y}\frac{\partial}{\partial\xi_y} \quad \text{and} \quad \frac{\partial}{\partial\eta}-\frac{\partial}{\partial x}\frac{\partial}{\partial\eta_x}-\frac{\partial}{\partial y}\frac{\partial}{\partial\eta_y} \tag{18}$$

The result is simply the Laplace equations for ξ and η, which have long been used as a generation system as noted above. Coordinate systems generated from the Laplace equations thus maximize smoothness. This minimization can be carried out in the transformed region by transforming the integral in Eq. (17) to

$$I_s = \int [(\nabla\xi)^2+(\nabla\eta)^2]\mathrm{d}x\mathrm{d}y = \int \frac{g_{11}+g_{22}}{\sqrt{g}}\,\mathrm{d}\xi\mathrm{d}\eta \tag{19}$$

and setting the application of the following operators on this integral to zero:

$$\frac{\partial}{\partial x}-\frac{\partial}{\partial\xi}\frac{\partial}{\partial x_\xi}-\frac{\partial}{\partial\eta}\frac{\partial}{\partial x_\eta} \quad \text{and} \quad \frac{\partial}{\partial y}-\frac{\partial}{\partial\xi}\frac{\partial}{\partial y_\xi}-\frac{\partial}{\partial\eta}\frac{\partial}{\partial y_\eta} \tag{20}$$

The off-diagonal metric coefficient g_{12} vanishes for an orthogonal system, and hence its magnitude is a measure of the departure from orthogonality. Since the metric coefficients are transformation properties, it seems appropriate to minimize the integral of g_{12} over the transformed region

$$I_0 \equiv \int g_{12}^2\,\mathrm{d}\xi\mathrm{d}\eta \tag{21}$$

Expressed in the physical region, this integral is

$$I_0 = \int (\nabla\xi\cdot\nabla\eta)^2 g^{3/2}\,\mathrm{d}x\mathrm{d}y \tag{22}$$

The Euler equations in either region then can be obtained by application of the appropriate operators to the corresponding integral as discussed above.

Orthogonality could also be controlled without the $g^{3/2}$ in the integral in the physical region. However, in Ref. 156, the inclusion of this term, which amounts to the use of g_{12} instead of $\nabla\xi\cdot\nabla\eta$ as the orthogonality measure, was found to relieve some problems with rounding errors. With g_{12} as the measure, large cells (where the Jacobian in large), are orthogonalized more effectively than are small cells. With $\nabla\xi\cdot\nabla\eta$ as the measure, the orthogonalization effect would apply equally to all cells. (Generalized Caucy-Riemann conditions have recently been used in Ref. 121 as the orthogonality measure in the integral I_0.)

Now the adaptive control of the grid spacing can be achieved if the product of the Jacobian (the cell area) and a positive weight function (to be specified) are kept constant over the field. This is accomplished by minimizing the integral of this nonzero product over the field

$$I_w \equiv \int w(x,y)\sqrt{g}\,\mathrm{d}x\mathrm{d}y \tag{23}$$

In the transformed plane this integral becomes

$$I_w = \int wg\,\mathrm{d}\xi\mathrm{d}\eta \tag{24}$$

The grid generation system is then formed by minimizing a weighted sum, I, of these integrals.

$$I \equiv I_s + \lambda_0\left(\frac{N}{A}\right)^2 I_0 + \lambda_w\left(\frac{N}{A}\right)^2\frac{I}{W}I_w \tag{25}$$

where N is the total number of points in the field, and W is the average weight function over the field

$$W = \frac{I}{A}\int w\mathrm{d}x\mathrm{d}y$$

(This scaling of the weighted sum is an extension of that given in Ref. 209.) The Euler equations for this variational problem, which will be the weighted sums of those for the individual integrals, form the system of partial differential equations from which the coordinate system is generated. These two equations have the form

$$\begin{aligned} b_1x_{\xi\xi}+b_2x_{\xi\eta}+b_3x_{\eta\eta}+a_1y_{\xi\xi}+a_2y_{\xi\eta}+a_3y_{\eta\eta} &= -(g/2w)\,w_x \\ a_1x_{\xi\xi}+a_2x_{\xi\eta}+a_3x_{\eta\eta}+c_1y_{\xi\xi}+c_2y_{\xi\eta}+c_3y_{\eta\eta} &= -(g/2w)\,w_y \end{aligned} \tag{26}$$

where the coefficients are quadratic functions of the first derivatives. Emphasis is varied among the sometimes competing features of smoothness, orthogonality, and adaptivity by the choice of the constants λ_0 and λ_w. For example, a large λ_0 will result in a grid that is very nearly orthogonal, at the cost of smoothness and adaptivity.

The weight function $w(x,y)$ is to be a function of some measure of the solution error or variation, so that the spacing will be reduced where the error or variation is large. Some rather spectacular results of the grid adapting to a reflected shock are given in Ref. 156. Here the magnitude of the pressure gradient was used in the weight function.

The extension of this approach to three dimensions follows immediately with the inclusion of the third gradient squared in the smoothness integral, the addition of two more analogous terms to the orthogonality integral, and the interpretation of $\sqrt{g}$ as the three-dimensional Jacobian, i.e., the cell volume.[156] Three separate orthogonality integrals, with three corresponding weighting factors, could be used in view of the difficulty of achieving complete orthogonality in three dimensions, as mentioned above.

This variational approach is also useful in one dimension to concentrate points on a boundary according to curvature. In this case, the orthogonality integral is dropped and the smoothness integral becomes

$$I_s = \int \xi_s^2 \mathrm{d}s = \int \frac{\mathrm{d}\xi}{s_\xi} \tag{27}$$

where s is the arc length along the boundary curve, and the weight function integral becomes

$$I_w = \int \frac{w}{\xi_s} \mathrm{d}s = \int w s_\xi^2 \mathrm{d}\xi \tag{28}$$

with the weight function proportional to the boundary curvature.

In one dimension, the weight function integral that is minimized in the variational approach of Ref. 18 becomes, in the transformed plane,

$$\int w x_\xi^2 \mathrm{d}\xi \tag{29}$$

Now if each grid point were connected to its neighbors on a straight line by springs with spring constants, $2w$, the total energy of the configuration would be exactly the same as the above integral, since x is the point separation and the integral performs the summation over the points. The equilibrium distribution of these points then would be that distribution for which this energy is a minimum. The Euler equation for this variation problem is simply $(2wx_\xi)_\xi = 0$, so that the equlibrium point distribution is that for which the product wx_ξ is constant along the line. The higher dimensional forms of the weight function integral in the variational approach of Ref. 18 are analogous to this, with area and volume in two and three dimensions, respectively, replacing the linear distance used in one dimension.

The spring analogy is introduced directly in Refs. 64 and 65. Both damping and smoothing of the resulting point distribution were used to control grid oscillation. The use of the first derivative with respect to the curvilinear coordinate in the method of Refs. 9 and 140 is equivalent to taking the weight function as f_x in the variational problem.

Note also that the Euler equation for this one-dimensional spring configuration can be written as

$$w x_{\xi\xi} + w_\xi x_\xi = 0 \tag{30}$$

which shows that the spring constant and the weight function of the variational formulation in one dimension are exactly equivalent to the control function of the one-dimensional elliptic generation system described above [Eq. (6)].

This concept of equidistributing a positive weight function has been used in a number of works on one-dimensional problems, such as Ref. 204, and in Ref. 45 this approach is used to cause a two-dimensional grid to adapt along one set of coordinate lines in a combustion problem. It may be preferable in many cases to subequidistribute the weight function so that the product wx_ξ, or its analog in higher dimensions, is bounded rather than constant over the grid. In Ref. 45 the constraint that the ratio of adjacent spacings be bounded above and below is added also. Here the adaptive grid eliminated oscillation in a combustion solution. The one-dimensional equidistribution case has a solution in quadratures, since if $wx_\xi = \text{const}$ we have

$$\xi(x) = \xi_1 + (\xi_2 - \xi_1)\left[\int_{x_1}^{x} w(\bar{x})\mathrm{d}\bar{x} \Big/ \int_{x_1}^{x_2} w(x)\mathrm{d}x\right] \tag{31}$$

where in general x is the arc length along a curve. This is essentially the approach used in the spring-analogy approach of Refs. 64 and 65, with the integrals being expressed as discrete summations. In Ref. 65 the weight function is taken as the magnitude of a gradient along the grid line bounded away from zero. Later applications of this approach are given in Ref. 44.

This approach is also followed in Ref. 3, using the magnitude of the second derivative with respect to the Cartesian coordinates as the weight function. This procedure is used to generate lines of points at each successive marching position from a preliminary flow solution on a coarse grid at that position in a parabolic flow solution. The use of the flow solution at the previous position to define the new grid line was found to lead to oscillation between the solution and the grid. Some results for flow in a channel with intermittent finite plates are given. Considerable improvement was noted with the adaptive grid.

Other recent applications of the one-dimensional form have been given in Refs. 123 and 2, as well as in several references discussed in Ref. 188, and further discussion appears also in Ref. 43, where it is noted that the cell Reynolds number problem is one of resolution and can be effectively removed with an adaptive grid with a fixed number of points. An example of adaption along an airfoil into a shock is given in Ref. 123. Two recent procedures for the application of equidistribution in two dimensions have been given in Refs. 51 and 52. Reference 51 involves a two-dimensional relaxation, while Ref. 52 applies successive alternatives of one-dimensional adaptions. Both of these procedures, as well as Ref. 2, use the solution curvature, i.e., second derivatives as well as the solution gradient in the weight function.

Moving Finite Element

The moving finite element method of Refs. 118 and 117 is a dynamically adaptive grid method in which the grid point locations are made additional dependent variables in a Galerkin formulation. The solution is expanded in piecewise-linear functions in terms of its values at the grid points and those of the grid point locations on each element. The residual is then required to be orthogonal to all the basis functions for both the solution grid point values. The grid point locations are thus obtained as a part of the finite element solution. An internodal viscosity is introduced to penalize the relative motion between the grid points. This does not penalize the absolute motion of the points. An internodal repulsive force is also introduced in Ref. 117 to maintain a minimum point separation. Both of these effects are strong, but of short range. A small, long-range attractive force was also introduced to keep the nodes more equally spaced in the absence of solution gradients. Small time steps are used in the initial development of the solution. The method has been extended to two dimensions. In Ref. 60 some tests were conducted with the moving finite element method which show that dispersion and dissipation are essentially eliminated. Square waves are convected exactly, as is shown in comparison with other methods. Numerous other examples are given for various equations. An order of magnitude increase in stability is realized over conventional methods. This work makes it clear that the key to reducing dispersion is adaptive grids.

An adaptive grid in a finite element solution is also used in Ref. 37, but the grid motion is not completely coupled with the solution. Rather, the grid is moved separately from the solution and only after certain time steps. The finite element formulation is based either on piecewise-linear or piecewise cubic Hermite approximations. In the latter case the first derivatives of the solution become additional dependent variables. These expansions are on a fixed grid. After a certain number of time steps the grid points are redistributed such that the product of the grid spacing to the p power and the magnitude of the pth derivative of the solution on the grid is equidistributed on the grid. The value of p is 1 for the linear approximation and 3 for the cubic. The pth derivative is obtained by finite differences. Some limitations were placed on the grid changes. The adaptive grid was found to be effective in eliminating oscillation.

Attraction Approach

A different approach to dynamically adaptive grids is taken in Refs. 9, 127,140, 7, and 8. Here, instead, generating new grid point locations through the solution of partial differential equations, the grid points move directly under the influence of mutual attraction or repulsion between points. This is accomplished by assigning to each grid an attraction proportional to the difference between the magnitude of some measure of error (or solution variation) and the average magnitude of this measure over all points. This causes points with values of this measure that exceed the average to attract other points, and thus to reduce the local spacing, while points with a measure less than the average will repel other points and hence increase the spacing. This attraction is attenuated by an inverse power of the point separation in the transformed field. The collective attraction of all other points is then made to induce a velocity for each grid point.

Reflections in boundaries in the transformed field are used to provide smooth grid motion near and on the boundaries. Since the transformed field is rectangular, this reflection is not complicated by the shape of the physical boundaries. A means of including terms that will induce rotational motion into the grid, also discussed in Ref. 9, has been devised to cause the grid lines to align with lines of high gradients such as shocks. Here the adaptive grid essentially eliminated the post-shock oscillations that occurred with the fixed grid.

In this approach, the magnitude of the first or second derivative has often been used as the variation measure which is driven toward a uniform distribution. Another measure used is the product of the first derivative with the derivative of the Cartesian coordinate with respect to the curvilinear, i.e., the spacing. The use of higher derivatives in the error expressions was found to result in noisy measures and erratic point motion. This arises because of the lack of accuracy in the difference representatives of these higher derivatives.

This procedure does not exercise any control over either the smoothness or orthogonality of the grid, so that distortion is possible. Collapse of points into each other, however, is impeded because the attraction will become repulsion as the points approach each other, since the measure which drives the motion will drop below the average as the spacing decreases. Collapse is impeded further by the fact that the grid velocities decrease with the spacing. It has been found necessary to apply some limits and some damping of the grid speeds to prevent grid oscillation and distortion. Smoothing through the addition of diffusion-like terms in the calculation of the grid movement from the grid speeds has also been used, cf., Ref. 8.

Since this procedure has all grid points moving to cause some measure to approach uniformity over the field, it can be considered an iterative approach to the solution of a variational problem to minimize the variation of this measure over the field. With the measure taken as $|u_\xi|$, as in Refs. 9 and 140, the equivalent weight function, w, of the one-dimensional variational problem discussed above is $w=|u_x|$. This occurs because the grid ceases to move when the measure is uniform, i.e., when the local value is equal to the average value everywhere. Therefore, the grid can be considered to move so as to minimize the variation in the measure over the field. Since the one-dimensional variational approach discussed above has the grid adjusting to minimize the variation of wx_ξ over the field, it follows that with the measure taken as $|u_\xi|=|u_x x_\xi|$, this attraction approach can be viewed as an implementation of the variational approach with the weight function $w=|u_x|$. With the measure taken as $|u_{\xi\xi}|$, as is also done in Refs. 9 and 140, we have

$$w=\left|u_{xx}x_\xi+u_x\frac{x_{\xi\xi}}{x_\xi}\right|$$

As noted above, these weight functions are also twice the equivalent spring constants in the spring analogy of Refs. 64 and 65, and also are the equivalent control functions in the one-dimensional version of the elliptic generating system given above [Eq. (6)].

A different, but somewhat related, approach is taken in Ref. 69 based on a chemical reaction analogy. Here each grid interval is taken to represent a species concentration, and the reaction rate constants are made dependent on the difference between a local error measure for one grid interval compared with another. Each grid interval then is coupled with every other grid interval through reaction rate equations, so that each integral grows at the expense of others, and vice versa. Some results for one and two dimensions are given. These rate constants also contain factors designed to limit the range of variation of the grid intervals.

Driving Measures

The evaluation of the driving measure of the error or solution variation is still a critical problem in all adaptive procedures. The truncation error expressions can be derived as noted above, although the algebra may be tedious, but these expressions involve higher order derivatives which may be very noisy when evaluated numerically, cf., Refs. 9 and 45. Therefore preference has usually been given to the use of solution variations involving lower derivatives.

In one dimension, if the grid is such that the same change in the dependent variable occurs between each pair of grid points, then the second and all higher derivatives with respect to the curvilinear coordinate will vanish, thus reducing the truncation error as noted above. This approach has been followed in Refs. 1, 9, and 57 for instance. Therefore, the first derivative with respect to the curvilinear coordinate is one logical choice for a driving measure. However, the use of this quantity alone is not completely satisfactory, since regions of uniform flow will become completely devoid of grid points, and the point distribution also will be depleted near solution extrema. Therefore, some influence of at least the second derivative needs to be included in the measure, and the measure should be bounded away from zero in uniform regions. Some of these considerations are discussed in Ref. 45.

When truncation error expressions are used for the measure, they should be expressed in terms of solution derivatives with respect to the Cartesian coordinates and the metric coefficients. The grid is to vary so as to reduce the error by changing the metric coefficients. In contrast, solution variations used as a driving measure to be driven to zero should be expressed in terms of solution derivatives with respect to the curvilinear coordinates since derivatives with respect to the Cartesian coordinates cannot be affected by the grid. Solution variations used as weight functions, as in the variational approach discussed above, however, are appropriately expressed in terms of the Cartesian derivatives, since the effort of the minimization is to concentrate points where the weight function is large.

The determination of appropriate measures for systems of equations is obviously more in doubt, and some sort of average influence of all the variables must be considered unless one variable is dominant or representative of all. In compressible flow applications with shocks, the density is a logical choice, although the Mach number,[65] internal energy,[65] or pressure[156] can serve also, while in boundary layers both the velocity magnitude and vorticity have been used.[57] In Ref. 3 individual point distributions are obtained for each variable and the most dense clustering is used. In Ref. 45 different grids were used for different equations, with interpolation between.

Moving Boundaries

Coordinate systems that are time-dependent because of boundary movement, or to follow some internal demarcation, are adaptive in the sense of following the delimiting feature rather than gradients or such. The incorporation of the moving grid into the solution algorithm, in general, is the same regardless of why the grid is moving. Several examples of grids following boundary motion are cited in Ref. 190.

Other examples are found in the free surface applications of Refs. 11, 77, and 208. In Refs. 144 and 145 the grid deforms to follow the solid-liquid interface in a melting problem. Other interface applications are given in Ref. 55. Reference 119 uses a grid which recedes to fit an advancing bullet in a muzzle. The grid moves to follow a pitching airfoil in Ref. 78 and to follow a bow shock in Ref. 146. Time-dependent free surface flows are treated in Refs. 11 and 77. Other results from the method of Ref. 77 appear in Ref. 208.

Various Applications

A variety of applications to physical problems are illustrated in Ref. 186 and more are cited in Ref. 190. In particular, Ref. 115 discusses application to solid mechanics, treating elastic torsion in shafts and the bending of thin elastic plates. Some examples from this work have been cited above. Reference 174 illustrates the application to external aerodynamics, Ref. 45 gives some applications to heat and mass transfer, and Ref. 101 discusses internal flows. Reference 92 covers some applications to flows in rivers and harbors. Spectral methods have been extended to include variable geometric coefficients, thus allowing these methods to be applied in the rectangular transformed plane.[131,114,176] Some compact difference schemes have recently been given in Ref. 13.

Conclusion

In conclusion, general three-dimensional regions are best treated by segmenting the field into subregions, with grids being generated in each subregion. The question of how much continuity to enforce at the interfaces between the subregions is still open, but means do exist for generating and using segmented grids with complete continuity and with no continuity, the former requiring iteration among the subregions, and the latter requiring interpolation in the region of the interfaces.

It is clearly necessary to give consideration to the rate of change of the grid line spacing, and also skewness, with regard to grid-induced error. Some error evaluation procedures are available, and this is an area of continuing effort.

Dynamically adaptive grids coupled with the physical solution evolving thereon can be expected to be the most promising area of research in the coming years. It has been noted that when the grid is properly positioned to resolve the solution gradients, most solution algorithms work well. Oscillations associated with cell Reynolds number and with shocks in fluid mechanics computations have been shown to be eliminated with adaptive grids. Even the numerical viscosity introduced by upwind differencing is reduced as the grid adapts to regions of large solution variation. Thus the ultimate answer to numerical solution of partial differential equations for computational fluid dynamics may well be dynamically adaptive grids, rather than more elaborate difference representations and solution methods.

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